

Question 1

Solve the equation:

$$2 \sin^2(x) - 3 \sin(x) + 1 = 0$$

for all $x \in [0, 2\pi]$.

Question 1 - Solution

We factor the equation:

$$2 \sin^2(x) - 3 \sin(x) + 1 = 0.$$

Looking for factors of $2 \cdot 1 = 2$ that add to -3 , we write:

$$(2 \sin(x) - 1)(\sin(x) - 1) = 0.$$

Setting each factor equal to zero:

$$2 \sin(x) - 1 = 0 \quad \text{or} \quad \sin(x) - 1 = 0.$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \sin(x) = 1.$$

Now solve for $x \in [0, 2\pi]$:

$$\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}.$$

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Therefore, the solutions are:

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}.$$

Question 2

Solve the equation

$$\cos(2x) = 2 \cos^2(x) - 1$$

for all $x \in [0, 2\pi]$, and verify the identity used.

Question 2 - Solution**Step 1: Recognize the identity.**

The right-hand side is a known double-angle identity:

$$\cos(2x) = 2 \cos^2(x) - 1$$

So this equation is always true:

$$\cos(2x) = \cos(2x)$$

Thus, it holds for all x .

But the problem is to solve:

$$\cos(2x) = 2 \cos^2(x) - 1$$

Since it holds for all x , this is an identity. So the equation is true for all $x \in [0, 2\pi]$

$$\boxed{x \in [0, 2\pi]}$$

Verification of identity:

Start with RHS:

$$2 \cos^2(x) - 1 = \cos(2x)$$

This is a standard double-angle identity for cosine:

$$\cos(2x) = 2 \cos^2(x) - 1 \Rightarrow \text{Identity verified.}$$

Question 3

Solve the equation

$$\sin(x) \cos(x) = \frac{1}{2}$$

for all $x \in [0, 2\pi]$.

Question 3 - Solution

We begin by using a known identity:

$$\sin(2x) = 2 \sin(x) \cos(x) \Rightarrow \sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

Substitute into the equation:

$$\frac{1}{2} \sin(2x) = \frac{1}{2} \Rightarrow \sin(2x) = 1$$

Now solve:

$$\sin(2x) = 1 \Rightarrow 2x = \frac{\pi}{2} + 2n\pi \quad \text{for integer } n \Rightarrow x = \frac{\pi}{4} + n\pi$$

Find solutions in $[0, 2\pi]$:

$$n = 0 \Rightarrow x = \frac{\pi}{4} \quad n = 1 \Rightarrow x = \frac{5\pi}{4} \quad n = 2 \Rightarrow x = \frac{9\pi}{4} > 2\pi \rightarrow \text{discard}$$

$$\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$$

Question 4

Solve the equation

$$\cos(2x) + \sin(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 4 - Solution**Step 1: Use a double-angle identity for $\cos(2x)$:**

We use:

$$\cos(2x) = 1 - 2\sin^2(x)$$

Substitute into the equation:

$$1 - 2\sin^2(x) + \sin(x) = 0 \Rightarrow -2\sin^2(x) + \sin(x) + 1 = 0$$

Multiply through by -1 to simplify:

$$2\sin^2(x) - \sin(x) - 1 = 0$$

Step 2: Solve the quadratic equation in $\sin(x)$:Let $u = \sin(x)$:

$$2u^2 - u - 1 = 0 \Rightarrow u = \frac{1 \pm \sqrt{1^2 + 4(2)(1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4}$$

So:

$$u = 1 \quad \text{or} \quad u = -\frac{1}{2}$$

Now solve $\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$
And $\sin(x) = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$ **Final answer:**

$x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
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Question 5

Solve the equation

$$\sin(x) + \sqrt{3}\cos(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 5 - Solution

We start by solving:

$$\sin(x) + \sqrt{3}\cos(x) = 0 \Rightarrow \sin(x) = -\sqrt{3}\cos(x)$$

Divide both sides by $\cos(x)$, assuming $\cos(x) \neq 0$:

$$\tan(x) = -\sqrt{3}$$

Now solve:

$$\tan(x) = -\sqrt{3} \Rightarrow x = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

But we want values in $[0, 2\pi]$

So find angles with tangent $= -\sqrt{3}$

The reference angle is $\frac{\pi}{3}$, and tangent is negative in quadrants II and IV

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}, \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

Also check if $\cos(x) = 0$

If $\cos(x) = 0 \Rightarrow \sin(x) = 0$ also must hold (from original equation)

But: - If $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ - At $x = \frac{\pi}{2}$: $\sin(x) = 1$, LHS $= 1 + 0 \neq 0$ - At $x = \frac{3\pi}{2}$: $\sin(x) = -1$, LHS $= -1 + 0 \neq 0$

So those are not valid. Final solution:

$$x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

Question 6

Solve the equation

$$2 \cos^2(x) - 1 = \sin(x)$$

for all $x \in [0, 2\pi]$.

Question 6 - Solution

We begin by using the Pythagorean identity:

$$\cos^2(x) = 1 - \sin^2(x)$$

Substitute into the equation:

$$2(1 - \sin^2(x)) - 1 = \sin(x) \Rightarrow 2 - 2\sin^2(x) - 1 = \sin(x) \Rightarrow -2\sin^2(x) - \sin(x) + 1 = 0$$

Multiply through by -1 :

$$2\sin^2(x) + \sin(x) - 1 = 0$$

Let $u = \sin(x)$. Then:

$$2u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$u = \frac{1}{2}, \quad u = -1$$

So:

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

Final solution:

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Question 7

Solve the equation

$$\sin(2x) = \sqrt{3} \cos(x)$$

for all $x \in [0, 2\pi]$.

Question 7 - Solution

Use the identity:

$$\sin(2x) = 2 \sin(x) \cos(x)$$

Substitute into the equation:

$$2 \sin(x) \cos(x) = \sqrt{3} \cos(x)$$

Subtract $\sqrt{3} \cos(x)$ from both sides:

$$2 \sin(x) \cos(x) - \sqrt{3} \cos(x) = 0 \Rightarrow \cos(x) (2 \sin(x) - \sqrt{3}) = 0$$

Now solve each factor:

****Case 1:**** $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

****Case 2:**** $2 \sin(x) - \sqrt{3} = 0 \Rightarrow \sin(x) = \frac{\sqrt{3}}{2}$

$$\sin(x) = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Final Solution:

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$
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Question 8

Solve the equation

$$\sin^2(x) - \cos(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 8 - Solution

We start with the identity:

$$\sin^2(x) = 1 - \cos^2(x)$$

Substitute into the equation:

$$1 - \cos^2(x) - \cos(x) = 0 \Rightarrow -\cos^2(x) - \cos(x) + 1 = 0 \Rightarrow \cos^2(x) + \cos(x) - 1 = 0$$

Let $u = \cos(x)$, then:

$$u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 + 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

So:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \approx 0.618, \quad \cos(x) = \frac{-1 - \sqrt{5}}{2} \approx -1.618$$

Only $\frac{-1 + \sqrt{5}}{2}$ is in the range $[-1, 1]$

Now solve:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \Rightarrow x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$$

Let's call $\theta = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$

Since cosine is positive in quadrants I and IV, and $\cos(x) > 0$, we get:

$$x = \theta, \quad x = 2\pi - \theta$$

So:

$$x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right), \quad x = 2\pi - \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$$

These are the exact answers in terms of inverse cosine.

Question 9

Solve the equation

$$\tan^2(x) - 3 \tan(x) + 2 = 0$$

for all $x \in [0, 2\pi)$.

Question 9 - Solution

Treat this as a quadratic in $\tan(x)$. Let $u = \tan(x)$:

$$u^2 - 3u + 2 = 0 \Rightarrow u = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm 1}{2} \Rightarrow u = 1, 2$$

So:

$$\tan(x) = 1 \quad \text{or} \quad \tan(x) = 2$$

Solve $\tan(x) = 1$:

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Solve $\tan(x) = 2$:

We must use the inverse tangent:

$$x = \tan^{-1}(2) \approx 1.107 \text{ radians}$$

Tangent is positive in quadrants I and III, so:

$$x = \tan^{-1}(2), \pi + \tan^{-1}(2)$$

Approximate values:

$$x \approx 1.107, 4.249$$

Final Answer:

$x = \frac{\pi}{4}, \frac{5\pi}{4}, \tan^{-1}(2), \pi + \tan^{-1}(2)$

Question 10

Solve the equation

$$\sec(x) = 2$$

for all $x \in [0, 2\pi]$.

Question 10 - Solution

Step 1: Recall the definition of secant:

$$\sec(x) = \frac{1}{\cos(x)} \Rightarrow \frac{1}{\cos(x)} = 2 \Rightarrow \cos(x) = \frac{1}{2}$$

Step 2: Solve $\cos(x) = \frac{1}{2}$ in $[0, 2\pi]$:

The cosine is $\frac{1}{2}$ at:

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$