

Question 1

Given the function $f(x) = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$, answer the following:

- (a) Determine the amplitude, period, phase shift, and vertical shift.
- (b) Find the maximum and minimum values of $f(x)$.
- (c) Sketch one full period of the graph, clearly labeling key points.
- (d) Determine the x -values where $f(x) = 1$ in the interval $[0, 2\pi]$.

Question 1 - Solution

(a) Identify parameters from $f(x) = 3 \sin(2x - \frac{\pi}{3}) + 1$:• Amplitude: $\boxed{3}$ • Period: $\frac{2\pi}{2} = \boxed{\pi}$

• Phase shift:

$$2x - \frac{\pi}{3} = 0 \Rightarrow x = \frac{\pi}{6} \Rightarrow \boxed{\frac{\pi}{6} \text{ to the right}}$$

• Vertical shift: $\boxed{+1}$

(b) Maximum and Minimum:

Since the sine function oscillates between -1 and 1 :

$$\text{Min of } f(x) = 3(-1) + 1 = -2, \quad \text{Max of } f(x) = 3(1) + 1 = 4$$

$$\boxed{\text{Max} = 4}, \quad \boxed{\text{Min} = -2}$$

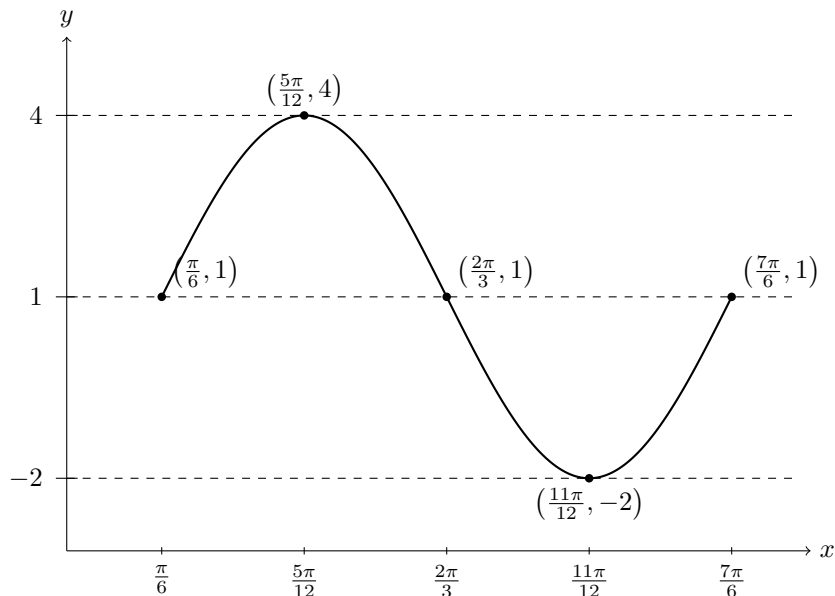
(c) Sketch one full period:

One full period has length π , starting at the phase shift:

$$x = \frac{\pi}{6} \quad \text{to} \quad x = \frac{7\pi}{6}$$

Key points over one period:

x	$f(x)$
$\frac{\pi}{6}$	1 (midline)
$\frac{5\pi}{12}$	4 (maximum)
$\frac{2\pi}{3}$	1 (midline)
$\frac{11\pi}{12}$	-2 (minimum)
$\frac{7\pi}{6}$	1 (midline)



(d) Solve $f(x) = 1$ on $[0, 2\pi]$:

$$3 \sin\left(2x - \frac{\pi}{3}\right) + 1 = 1 \Rightarrow \sin\left(2x - \frac{\pi}{3}\right) = 0$$

$$2x - \frac{\pi}{3} = n\pi \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{6}$$

Values in $[0, 2\pi]$:

$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Question 2

Let

$$g(x) = 2 \tan\left(x - \frac{\pi}{4}\right) + 1$$

- (a) State the period, phase shift, vertical shift, and asymptotes of $g(x)$.
- (b) Sketch one period of the graph, labeling asymptotes and intercepts.
- (c) Find the value(s) of $x \in [0, 2\pi]$ such that $g(x) = 3$.

Question 2 - Solution

(a) **Analyze** $g(x) = 2 \tan\left(x - \frac{\pi}{4}\right) + 1$

- Period: π
- Phase shift: $\frac{\pi}{4}$ to the right
- Vertical shift: $+1$

Asymptotes occur when:

$$x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \Rightarrow x = \frac{3\pi}{4} + k\pi$$

So one period lies on:

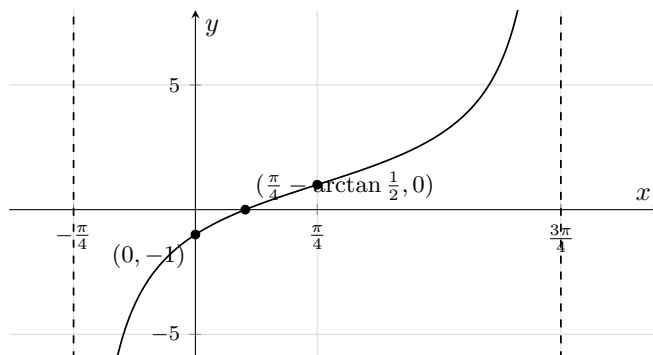
$$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$$

- All asymptotes: $x = 3\pi/4 + k\pi, \quad k \in \mathbb{Z}$.

(b) **Sketch one period**The intercepts in this period are $(0, -1)$ and $(\pi/4 - \arctan(1/2), 0)$.

Key features:

- Left asymptote: $x = -\frac{\pi}{4}$
- Right asymptote: $x = \frac{3\pi}{4}$
- Midpoint (inflection): $x = \frac{\pi}{4}, \quad g\left(\frac{\pi}{4}\right) = 1$

(c) **Solve** $g(x) = 3$:

$$2 \tan\left(x - \frac{\pi}{4}\right) + 1 = 3 \Rightarrow \tan\left(x - \frac{\pi}{4}\right) = 1$$

$$x - \frac{\pi}{4} = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{2} + n\pi$$

Values in $[0, 2\pi]$:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Question 3

Consider the function

$$f(x) = \cos(2x) - \sin(2x)$$

- (a) Write $f(x)$ as a single trigonometric function of the form $A \cos(2x + \phi)$.
- (b) Determine the amplitude and phase shift of the resulting function.
- (c) Find the maximum and minimum values of $f(x)$.
- (d) Determine all $x \in [0, 2\pi]$ such that $f(x) = 0$.

Question 3 - Solution

(a) Write $f(x)$ as a single cosine function:

We use the identity:

$$A \cos(2x + \phi) = A \cos \phi \cos(2x) - A \sin \phi \sin(2x)$$

Match coefficients with:

$$\cos(2x) - \sin(2x)$$

This gives the system:

$$A \cos \phi = 1, \quad A \sin \phi = 1$$

Square and add:

$$A^2(\cos^2 \phi + \sin^2 \phi) = 1^2 + 1^2 = 2 \Rightarrow A = \sqrt{2}$$

Now:

$$\cos \phi = \frac{1}{\sqrt{2}}, \quad \sin \phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$$

So:

$$f(x) = \sqrt{2} \cos\left(2x + \frac{\pi}{4}\right)$$

(b) Amplitude and Phase Shift:

From the rewritten form:

- Amplitude: $\boxed{\sqrt{2}}$
- Phase shift: Solve $2x + \frac{\pi}{4} = 0 \Rightarrow x = -\frac{\pi}{8}$

So the graph is shifted $\boxed{\frac{\pi}{8}}$ units to the left.

(c) Maximum and Minimum Values:

Since cosine ranges between -1 and 1 :

$$\text{Maximum} = \sqrt{2}, \quad \text{Minimum} = -\sqrt{2}$$

(d) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$\cos(2x) - \sin(2x) = 0 \Rightarrow \cos(2x) = \sin(2x) \Rightarrow \tan(2x) = 1$$

$$2x = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{8} + \frac{n\pi}{2}$$

Find values in $[0, 2\pi]$:

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

Answer:

$$\boxed{x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}}$$

Question 4

Let $f(x) = \sin(x)\cos(x) + \cos^2(x)$

- (a) Express $f(x)$ in terms of a single trigonometric function or a simpler expression.
- (b) Find the maximum and minimum values of $f(x)$ on the interval $[0, 2\pi]$.
- (c) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 1$.

Question 4 - Solution

Simplify. Double-angle identities give

$$f(x) = \frac{1}{2} + \frac{\sqrt{2}}{2} \sin(2x + \pi/4).$$

The sine attains both -1 and 1 on the given interval, so

$$f_{\min} = \frac{1 - \sqrt{2}}{2}, \quad f_{\max} = \frac{1 + \sqrt{2}}{2}.$$

Solve without losing roots. Subtracting 1 from the original expression gives

$$f(x) - 1 = \sin x \cos x - \sin^2 x = \sin x(\cos x - \sin x).$$

Thus $\sin x = 0$ or $\cos x = \sin x$. Restricting both families to $[0, 2\pi]$ gives

$$x = 0, \pi/4, \pi, 5\pi/4, 2\pi.$$

Each listed value makes one factor zero, so each satisfies the original equation.

Question 5

Let $f(x) = \frac{\sin(x)}{1+\cos(x)}$

- (a) Simplify the expression $f(x)$ using trigonometric identities.
- (b) Determine all $x \in (0, 2\pi)$ for which $f(x) = 1$.
- (c) Determine the domain of $f(x)$.
- (d) Determine whether $f(x)$ is an even function, odd function, or neither.

Question 5 - Solution

Simplify with the domain retained. Half-angle identities give

$$\boxed{f(x) = \tan(x/2)}, \quad x \neq (2k+1)\pi, \quad k \in \mathbb{Z}.$$

Indeed $\sin x = 2 \sin(x/2) \cos(x/2)$ and $1 + \cos x = 2 \cos^2(x/2)$.

The alternative $\csc x - \cot x$ is valid only when $\sin x \neq 0$ and loses valid inputs such as $x = 0$.

Solve. $\tan(x/2) = 1$ implies $x = \pi/2 + 2k\pi$, so on $(0, 2\pi)$,

$$\boxed{x = \pi/2}.$$

Domain and parity. The original denominator vanishes precisely at odd multiples of π , hence

$$\boxed{D = \mathbb{R} \setminus \{(2k+1)\pi : k \in \mathbb{Z}\}}.$$

This domain is symmetric about zero, and $f(-x) = -f(x)$ on it. Therefore $\boxed{f \text{ is odd}}$.

Question 6

Let $f(x) = \sin(x) + \sin(3x)$

- (a) Use a trigonometric identity to write $f(x)$ as a product.
- (b) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 0$.
- (c) Sketch the graph of $f(x)$ over one full period.

Question 6 - Solution

(a) Use sum-to-product identities:

We apply:

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

Let $A = 3x$, $B = x$:

$$f(x) = \sin(x) + \sin(3x) = 2 \sin \left(\frac{3x+x}{2} \right) \cos \left(\frac{3x-x}{2} \right) = 2 \sin(2x) \cos(x)$$

$$f(x) = 2 \sin(2x) \cos(x)$$

(b) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$2 \sin(2x) \cos(x) = 0 \Rightarrow \sin(2x) = 0 \quad \text{or} \quad \cos(x) = 0$$

Solve $\sin(2x) = 0$:

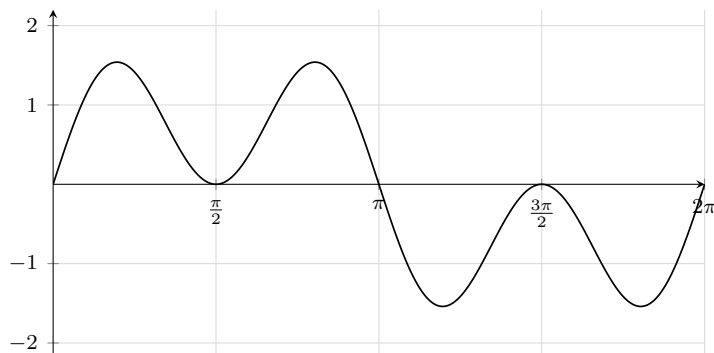
$$2x = n\pi \Rightarrow x = \frac{n\pi}{2} \Rightarrow x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Solve $\cos(x) = 0$:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Union of solutions:

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

(c) Sketch the graph of $f(x) = \sin(x) + \sin(3x)$ The graph is periodic with the least common multiple of the periods of $\sin(x)$ and $\sin(3x)$.Period of $\sin(x)$: 2π Period of $\sin(3x)$: $\frac{2\pi}{3}$ LCM of 2π and $\frac{2\pi}{3}$ is 2π So sketch one full period over $[0, 2\pi]$. Use the product form $f(x) = 2 \sin(2x) \cos(x)$ to help identify:Zeros: Where either $\sin(2x) = 0$ or $\cos(x) = 0$ Amplitude varies due to the product of two trig functionsImportant points: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and midpoints between for peaks/troughs

Question 7

Let

$$f(x) = \frac{1 + \cos(2x)}{\sin(2x)}$$

- (a) Simplify the expression as much as possible.
- (b) Find all $x \in (0, \pi)$ for which $f(x) = 2$.
- (c) Determine the domain of $f(x)$ over $[0, 2\pi]$.

Question 7 - Solution

Simplify. Where the original function is defined,

$$f(x) = \frac{2 \cos^2 x}{2 \sin x \cos x} = \cot x.$$

The cancellation does not restore points where $\sin(2x) = 0$.

Solve. On $(0, \pi)$, $\cot x = 2$ gives

$$x = \arctan(1/2) \approx 0.4636.$$

This point is in the original domain.

Domain. All multiples of $\pi/2$ must be excluded. On $[0, 2\pi]$ the domain is

$$(0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi).$$

Question 8

Let

$$f(x) = \frac{\cos(x)}{1 - \sin(x)}$$

- (a) Simplify the expression $f(x)$ using trigonometric identities.
- (b) Determine all values of $x \in (0, 2\pi)$ for which $f(x) = 1$.
- (c) Determine whether $f(x)$ is even, odd, or neither.

Question 8 - Solution

Simplification and domain. The original function is defined for $x \neq \pi/2 + 2k\pi$. Where $\cos x \neq 0$, rationalization gives

$$f(x) = \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$$

At $x = 3\pi/2 + 2k\pi$, the original function has value 0, although this alternative expression is undefined.

Solve on the requested interval. If $f(x) = 1$, then $\cos x + \sin x = 1$, with $\sin x \neq 1$.

Using $\cos x = 1 - \sin x$ in $\cos^2 x + \sin^2 x = 1$ gives

$$2 \sin x (\sin x - 1) = 0.$$

The case $\sin x = 1$ is excluded. The other case requires $\sin x = 0$ and $\cos x = 1$, hence $x = 2k\pi$.

Neither endpoint belongs to $(0, 2\pi)$, so

There are no solutions in $(0, 2\pi)$.

Parity. The domain is not symmetric about zero: $-\pi/2$ is allowed but $\pi/2$ is not.

f is neither even nor odd.

Question 9

Let

$$f(x) = \tan(x) - \sec(x)$$

- (a) Show that $f(x) = \frac{\sin(x)-1}{\cos(x)}$
- (b) Determine the values of $x \in (0, 2\pi)$ for which $f(x) = 0$
- (c) Analyze the behavior of $f(x)$ near $x = \frac{\pi}{2}$

Question 9 - Solution

(a) Simplify $f(x) = \tan(x) - \sec(x)$:

We write both terms in terms of sine and cosine:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}$$

Then:

$$f(x) = \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} = \frac{\sin(x) - 1}{\cos(x)}$$

$$f(x) = \frac{\sin(x) - 1}{\cos(x)}$$

(b) Solve $f(x) = 0 \Rightarrow \frac{\sin(x)-1}{\cos(x)} = 0$

This occurs when the numerator is 0 (and denominator is not 0):

$$\sin(x) - 1 = 0 \Rightarrow \sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$$

But at $x = \frac{\pi}{2}$, $\cos(x) = 0$, so $f(x)$ is undefined.Therefore, there is **no solution** to $f(x) = 0$ on $(0, 2\pi)$

No solution in $(0, 2\pi)$

(c) Behavior near $x = \frac{\pi}{2}$ As $x \rightarrow \frac{\pi}{2}^-$, we have: - $\sin(x) \rightarrow 1$ - $\cos(x) \rightarrow 0^+$ - So: $f(x) \rightarrow \frac{1-1}{0^+} = 0$ As $x \rightarrow \frac{\pi}{2}^+$, we have: - $\sin(x) \rightarrow 1$ - $\cos(x) \rightarrow 0^-$ - So: $f(x) \rightarrow \frac{1-1}{0^-} = 0$ But if we take x very close to $\frac{\pi}{2}$, with small deviation, say:

$$f(x) = \frac{\sin(x) - 1}{\cos(x)} \Rightarrow \text{Numerator} \approx 0, \text{ Denominator} \rightarrow 0$$

Apply L'Hôpital's Rule:

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0$$

So:

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = 0$$

But since the function is undefined at $x = \frac{\pi}{2}$, the limit exists but the function is discontinuous there.

Question 10

Define the function:

$$f(x) = \frac{1 - \cos(2x)}{\sin(x)}$$

- (a) Simplify $f(x)$ using appropriate identities.
- (b) Determine the values of $x \in (0, \pi)$ for which $f(x) = 2 \sin(x)$.
- (c) State the domain of $f(x)$ on $[0, 2\pi]$.

Question 10 - Solution**(a) Simplify using identities:**

Using the identity

$$1 - \cos(2x) = 2 \sin^2(x),$$

we obtain

$$f(x) = \frac{2 \sin^2(x)}{\sin(x)}.$$

For $\sin(x) \neq 0$, this simplifies to

$$f(x) = 2 \sin(x).$$

$$f(x) = 2 \sin(x) \quad \text{for } \sin(x) \neq 0$$

(b) Solve $f(x) = 2 \sin(x)$

From part (a), the equation

$$f(x) = 2 \sin(x)$$

is an identity for all x where $f(x)$ is defined.

However, the original function is undefined when

$$\sin(x) = 0 \Rightarrow x = 0, \pi.$$

Since the interval is restricted to $(0, \pi)$, neither excluded value lies in the interval. Therefore, all values in $(0, \pi)$ satisfy the equation.

$$\text{Solution set: } (0, \pi) \quad (\text{excluded: none within the interval})$$

(c) Domain of $f(x) = \frac{1 - \cos(2x)}{\sin(x)}$ on $[0, 2\pi]$

The function is undefined when the denominator is zero:

$$\sin(x) = 0 \Rightarrow x = 0, \pi, 2\pi.$$

Removing these values from the interval $[0, 2\pi]$, the domain is

$$(0, \pi) \cup (\pi, 2\pi).$$

Excluded values: $x = 0, \pi, 2\pi$