

Question 1

Let the function $f(x) = \frac{2x + 5}{x - 3}$.

- (a) Show that f is one-to-one.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

Question 1 - Solution

(a) **Show that f is one-to-one:**

A function is one-to-one if it passes the Horizontal Line Test, or algebraically if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Suppose:

$$\frac{2x_1 + 5}{x_1 - 3} = \frac{2x_2 + 5}{x_2 - 3}$$

Cross-multiplying:

$$(2x_1 + 5)(x_2 - 3) = (2x_2 + 5)(x_1 - 3)$$

Expanding both sides:

$$2x_1x_2 - 6x_1 + 5x_2 - 15 = 2x_1x_2 - 6x_2 + 5x_1 - 15$$

Subtract $2x_1x_2$ and -15 from both sides:

$$-6x_1 + 5x_2 = -6x_2 + 5x_1 \Rightarrow 5x_2 + 6x_2 = 5x_1 + 6x_1 \Rightarrow 11x_2 = 11x_1 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{2x + 5}{x - 3}$, and solve for x :

$$y(x - 3) = 2x + 5 \Rightarrow yx - 3y = 2x + 5 \Rightarrow yx - 2x = 3y + 5 \Rightarrow x(y - 2) = 3y + 5 \Rightarrow x = \frac{3y + 5}{y - 2}$$

Now swap x and y :

$$f^{-1}(x) = \frac{3x + 5}{x - 2}$$

(c) **Domain and Range:**

For $f(x) = \frac{2x + 5}{x - 3}$: - Denominator cannot be 0 $x \neq 3$ - So domain of f is: $\boxed{(-\infty, 3) \cup (3, \infty)}$

Range: Find y values that cannot occur. From inverse function: $f^{-1}(x) = \frac{3x + 5}{x - 2} \Rightarrow x \neq 2$

So range of f is $\boxed{(-\infty, 2) \cup (2, \infty)}$

Domain and Range:

Domain of $f(x) : (-\infty, 3) \cup (3, \infty)$

Range of $f(x) : (-\infty, 2) \cup (2, \infty)$

Domain of $f^{-1}(x) : (-\infty, 2) \cup (2, \infty)$

Range of $f^{-1}(x) : (-\infty, 3) \cup (3, \infty)$

Question 2

Let $f(x) = \ln(x - 1)$, where $x > 1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of $f(x)$ and $f^{-1}(x)$.
- (c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 2 - Solution

(a) Find the inverse of $f(x) = \ln(x - 1)$:

Let $y = \ln(x - 1)$

Exponentiate both sides:

$$e^y = x - 1 \Rightarrow x = e^y + 1$$

Now swap x and y to get the inverse:

$$f^{-1}(x) = e^x + 1$$

Answer: $f^{-1}(x) = e^x + 1$

(b) **Domain and Range:**

For $f(x) = \ln(x - 1)$:

Domain: $x > 1 \Rightarrow (1, \infty)$, Range of $\ln(x - 1)$ is all real numbers: $(-\infty, \infty)$

So for the inverse:

Domain of $f^{-1}(x)$: $(-\infty, \infty)$, Range of $f^{-1}(x) = e^x + 1$: since $e^x > 0$, we get $f^{-1}(x) > 1 \Rightarrow (1, \infty)$

Summary:

Domain of $f(x) = (1, \infty)$

Range of $f(x) = (-\infty, \infty)$

Domain of $f^{-1}(x) = (-\infty, \infty)$

Range of $f^{-1}(x) = (1, \infty)$

(c) **Verify compositions:**

$$f(f^{-1}(x)) = \ln((e^x + 1) - 1) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = e^{\ln(x-1)} + 1 = x - 1 + 1 = x$$

Both identities verified.

Question 3

Let $f(x) = \frac{x}{x+1}$, where $x \neq -1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (c) Verify the identity $f(f^{-1}(x)) = x$.

Question 3 - Solution

(a) Find the inverse of $f(x) = \frac{x}{x+1}$:

Let $y = \frac{x}{x+1}$. Solve for x :

Multiply both sides by $x+1$:

$$y(x+1) = x \Rightarrow yx + y = x \Rightarrow yx - x = -y \Rightarrow x(y-1) = -y \Rightarrow x = \frac{-y}{y-1}$$

Now switch x and y :

$$f^{-1}(x) = \frac{-x}{x-1}$$

Answer: $f^{-1}(x) = \frac{-x}{x-1}$

(b) Domain and Range:

For $f(x) = \frac{x}{x+1}$: Domain: $x \neq -1$, As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 1$, Solve $\frac{x}{x+1} = 1 \Rightarrow x = x+1 \Rightarrow 0 = 1 \rightarrow$ contradiction
So $f(x) \neq 1$

Domain: $(-\infty, -1) \cup (-1, \infty)$ **Range:** $(-\infty, 1) \cup (1, \infty)$

Now for $f^{-1}(x) = \frac{-x}{x-1}$:

Domain excludes $x = 1$, since denominator is zero Range excludes $y = -1$, since solving $\frac{-x}{x-1} = -1 \Rightarrow x = \frac{1}{2}$, which contradicts range of original function

Domain of $f^{-1}(x) = (-\infty, 1) \cup (1, \infty)$ **Range of** $f^{-1}(x) = (-\infty, -1) \cup (-1, \infty)$

(c) Verify:

$$f(f^{-1}(x)) = f\left(\frac{-x}{x-1}\right) = \frac{\frac{-x}{x-1}}{\frac{-x}{x-1} + 1} = \frac{\frac{-x}{x-1}}{\frac{-x + (x-1)}{x-1}} = \frac{\frac{-x}{x-1}}{\frac{-1}{x-1}} = x$$

Verified: $f(f^{-1}(x)) = x$

Question 4

Suppose $f(x) = \frac{1}{x^2 + 1}$.

- (a) Determine whether $f(x)$ has an inverse function. Justify.
- (b) If not, restrict the domain of f so that it becomes one-to-one and then find the inverse on that restricted domain.
- (c) State the domain and range of the inverse function you found.

Question 4 - Solution

(a) Does $f(x) = \frac{1}{x^2 + 1}$ have an inverse?

No. Its natural domain is $(-\infty, \infty)$, since $x^2 + 1$ is never zero. To have an inverse function, f must be **one-to-one**: different inputs must give different outputs.

Horizontal line test: Imagine drawing horizontal lines across the graph. Each horizontal line represents a fixed output y . If even one such line crosses the graph more than once, then different inputs give the same output, so the function is not one-to-one.

Testing points: We can show this without drawing the graph by choosing two different inputs and calculating their outputs. Try $x = 1$ and $x = -1$, since squaring either gives 1:

$$f(1) = \frac{1}{1^2 + 1} = \frac{1}{2}, \quad f(-1) = \frac{1}{(-1)^2 + 1} = \frac{1}{2}.$$

Thus, the horizontal line $y = \frac{1}{2}$ meets the graph at both $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$. An inverse would have to send $\frac{1}{2}$ back to both 1 and -1 , but a function can give only one output for each input.

So, f fails the horizontal line test and has no inverse function on $(-\infty, \infty)$. One pair of different inputs with the same output is enough to prove this; testing a few points with different outputs would not prove that a function is one-to-one everywhere.

(b) **Restrict domain to make f one-to-one:**

Choose only nonnegative inputs. As x increases on $[0, \infty)$, the denominator $x^2 + 1$ increases, so $f(x)$ strictly decreases. Therefore, no output repeats, and the restricted function is one-to-one. We use:

$$x \in [0, \infty)$$

Let $y = \frac{1}{x^2 + 1}$, with $x \geq 0$.
Solve for x :

$$\begin{aligned} y(x^2 + 1) &= 1 \\ x^2 + 1 &= \frac{1}{y} \\ x^2 &= \frac{1}{y} - 1 = \frac{1 - y}{y} \\ x &= \sqrt{\frac{1 - y}{y}}. \end{aligned}$$

We take only the nonnegative square root because our restricted domain requires $x \geq 0$. Now switch x and y to express the inverse using x as its input:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad \text{for } 0 < x \leq 1$$

(c) **Domain and range of the inverse function:**

The restricted function has domain $[0, \infty)$. Its largest output is $f(0) = 1$. As x increases, its outputs approach 0 but never equal 0, so its range is $(0, 1]$.

An inverse reverses inputs and outputs. Therefore, the original range becomes the inverse's domain, and the original domain becomes the inverse's range:

$$\text{Domain of } f^{-1}(x) = (0, 1], \quad \text{Range of } f^{-1}(x) = [0, \infty)$$

Answer:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad x \in (0, 1]$$

Question 5

Let $f(x) = \frac{3x - 4}{5x + 2}$, where $x \neq -\frac{2}{5}$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f(x)$ and $f^{-1}(x)$.

Question 5 - Solution

(a) **Show that f is one-to-one:**

Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{3x_1 - 4}{5x_1 + 2} = \frac{3x_2 - 4}{5x_2 + 2}$$

Cross-multiplying:

$$(3x_1 - 4)(5x_2 + 2) = (3x_2 - 4)(5x_1 + 2)$$

Expanding both sides:

$$15x_1x_2 + 6x_1 - 20x_2 - 8 = 15x_1x_2 + 6x_2 - 20x_1 - 8$$

Cancel $15x_1x_2$ and -8 :

$$6x_1 - 20x_2 = 6x_2 - 20x_1 \Rightarrow 6x_1 + 20x_1 = 6x_2 + 20x_2 \Rightarrow 26x_1 = 26x_2 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{3x-4}{5x+2}$

Multiply both sides:

$$y(5x + 2) = 3x - 4 \Rightarrow 5xy + 2y = 3x - 4 \Rightarrow 5xy - 3x = -2y - 4 \Rightarrow x(5y - 3) = -2y - 4 \Rightarrow x = \frac{-2y - 4}{5y - 3}$$

Now swap x and y :

$$f^{-1}(x) = \frac{-2x - 4}{5x - 3}$$

(c) **Domain and Range:**

The only restriction on the domain of f is $5x + 2 \neq 0 \Rightarrow x \neq -\frac{2}{5}$

The inverse is undefined when the denominator is 0:

$$5x - 3 \neq 0 \Rightarrow x \neq \frac{3}{5}$$

So:

$$\text{Domain of } f(x) : \boxed{\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, \infty\right)}$$

$$\text{Range of } f(x) : \boxed{\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)}$$

$$\text{Domain of } f^{-1}(x) : \boxed{\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)}$$

$$\text{Range of } f^{-1}(x) : \boxed{\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, \infty\right)}$$

Question 6

Let $f(x) = \sqrt{2x - 1}$ with domain $x \geq \frac{1}{2}$.

- (a) Show that f is one-to-one and has an inverse.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 6 - Solution**(a) One-to-One Check:**

Since $f(x) = \sqrt{2x-1}$ is a square root function with a linear expression inside and domain $x \geq \frac{1}{2}$, it is strictly increasing. Therefore, it passes the horizontal line test and is one-to-one.

(b) Find the inverse:

Let $y = \sqrt{2x-1}$. Solve for x :

$$y^2 = 2x - 1 \Rightarrow 2x = y^2 + 1 \Rightarrow x = \frac{y^2 + 1}{2}$$

Now switch x and y :

$$f^{-1}(x) = \frac{x^2 + 1}{2}$$

Answer: $\boxed{f^{-1}(x) = \frac{x^2 + 1}{2}}$

(c) Domain and Range of Inverse:

Since original domain is $x \geq \frac{1}{2}$, the range is:

$$f(x) = \sqrt{2x-1} \geq 0 \Rightarrow \text{Range of } f = [0, \infty)$$

So: - Domain of $f^{-1}(x) = [0, \infty)$ - Range of $f^{-1}(x) = [\frac{1}{2}, \infty)$

(d) Verifying composition:

$$f(f^{-1}(x)) = f\left(\frac{x^2 + 1}{2}\right) = \sqrt{2 \cdot \frac{x^2 + 1}{2} - 1} = \sqrt{x^2 + 1 - 1} = \sqrt{x^2} = x \quad (\text{since } x \geq 0)$$

$$f^{-1}(f(x)) = f^{-1}(\sqrt{2x-1}) = \frac{(\sqrt{2x-1})^2 + 1}{2} = \frac{2x - 1 + 1}{2} = x$$

Both identities verified.

Question 7

Let $f(x) = \frac{6}{x+2} - 1$, with domain $x \neq -2$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of both $f(x)$ and $f^{-1}(x)$.

Question 7 - Solution

(a) Show $f(x) = \frac{6}{x+2} - 1$ is one-to-one:

Let $f(x_1) = f(x_2)$. Then:

$$\frac{6}{x_1+2} - 1 = \frac{6}{x_2+2} - 1 \Rightarrow \frac{6}{x_1+2} = \frac{6}{x_2+2} \Rightarrow x_1+2 = x_2+2 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) Find the inverse function:

Let $y = \frac{6}{x+2} - 1$

Solve for x :

$$y+1 = \frac{6}{x+2} \Rightarrow x+2 = \frac{6}{y+1} \Rightarrow x = \frac{6}{y+1} - 2$$

Now switch x and y :

$$f^{-1}(x) = \frac{6}{x+1} - 2$$

Answer: $f^{-1}(x) = \frac{6}{x+1} - 2$

(c) Domain and Range:

Original function $f(x) = \frac{6}{x+2} - 1$

- Undefined when $x+2=0 \Rightarrow x=-2$ - Domain of f : $(-\infty, -2) \cup (-2, \infty)$

Asymptote at $x=-2$, and horizontal asymptote $y=-1$ - Range of f : $(-\infty, -1) \cup (-1, \infty)$

Inverse: - Domain of f^{-1} : same as range of f : $(-\infty, -1) \cup (-1, \infty)$ - Range of f^{-1} : same as domain of f : $(-\infty, -2) \cup (-2, \infty)$

Question 8

Let $f(x) = \frac{x+3}{x-1}$, where $x \neq 1$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$.

Question 8 - Solution

(a) Show that f is one-to-one:Let $f(x_1) = f(x_2)$:

$$\frac{x_1 + 3}{x_1 - 1} = \frac{x_2 + 3}{x_2 - 1} \Rightarrow (x_1 + 3)(x_2 - 1) = (x_2 + 3)(x_1 - 1)$$

Expanding both sides:

$$x_1x_2 - x_1 + 3x_2 - 3 = x_1x_2 - x_2 + 3x_1 - 3$$

Subtract x_1x_2 and -3 from both sides:

$$-x_1 + 3x_2 = -x_2 + 3x_1 \Rightarrow 3x_2 + x_2 = 3x_1 + x_1 \Rightarrow 4x_2 = 4x_1 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) Find the inverse:

$$\text{Let } y = \frac{x + 3}{x - 1}$$

Multiply both sides:

$$y(x - 1) = x + 3 \Rightarrow yx - y = x + 3 \Rightarrow yx - x = y + 3 \Rightarrow x(y - 1) = y + 3 \Rightarrow x = \frac{y + 3}{y - 1}$$

Switch x and y :

$$f^{-1}(x) = \frac{x + 3}{x - 1}$$

So:

$$\boxed{f^{-1}(x) = \frac{x + 3}{x - 1}}$$

Interestingly, $f^{-1}(x) = f(x)$. The function is its own inverse.

(c) Domain and Range:

From $f(x) = \frac{x + 3}{x - 1}$, we see that: - Domain: $x \neq 1 \Rightarrow \boxed{(-\infty, 1) \cup (1, \infty)}$ - As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and $f(x) \neq 1$ - So range: $\boxed{(-\infty, 1) \cup (1, \infty)}$

The same applies to $f^{-1}(x)$, so: - Domain of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$ - Range of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$

(d) Verify:

$$f(f^{-1}(x)) = f\left(\frac{x + 3}{x - 1}\right) = \frac{\left(\frac{x + 3}{x - 1}\right) + 3}{\left(\frac{x + 3}{x - 1}\right) - 1} = \frac{\frac{x + 3 + 3(x - 1)}{x - 1}}{\frac{x + 3 - (x - 1)}{x - 1}} = \frac{\frac{x + 3 + 3x - 3}{x - 1}}{\frac{x + 3 - x + 1}{x - 1}} = \frac{\frac{4x}{x - 1}}{\frac{4}{x - 1}} = x$$

Verified: $f(f^{-1}(x)) = x$

Question 9

Let $f(x) = \ln(x - 2)$, with domain $x > 2$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) Determine the domain and range of f and f^{-1} .
- (c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 9 - Solution**(a) Find the inverse:**Let $y = \ln(x - 2)$ Solve for x :

$$x - 2 = e^y \Rightarrow x = e^y + 2$$

Now switch x and y :

$$f^{-1}(x) = e^x + 2$$

Answer: $f^{-1}(x) = e^x + 2$ **(b) Domain and Range:**For $f(x) = \ln(x - 2)$: - Domain: $x > 2 \Rightarrow (2, \infty)$ - Range: $\ln(x - 2) \rightarrow -\infty$ as $x \rightarrow 2^+$, and $\rightarrow \infty$ as $x \rightarrow \infty$ Range: $(-\infty, \infty)$ For $f^{-1}(x) = e^x + 2$: - Domain: $(-\infty, \infty)$ - Range: $e^x + 2 > 2 \Rightarrow (2, \infty)$ **(c) Verify compositions:**

$$f(f^{-1}(x)) = f(e^x + 2) = \ln((e^x + 2) - 2) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = f^{-1}(\ln(x - 2)) = e^{\ln(x - 2)} + 2 = x - 2 + 2 = x$$

Verified: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Question 10

Let

$$f(x) = \frac{\sqrt{x+4}}{x-1}$$

- (a) Determine the domain of $f(x)$.
- (b) Is f one-to-one on its domain? Justify your answer.
- (c) If possible, find $f^{-1}(x)$. If not, restrict the domain appropriately and find the inverse on that domain.

Question 10 - Solution

Domain. The square root and denominator require

$$D = [-4, 1) \cup (1, \infty).$$

One-to-one. For $x > -4$, $x \neq 1$,

$$f'(x) = \frac{-(x+9)}{2\sqrt{x+4}(x-1)^2} < 0.$$

The left branch decreases from 0 to $-\infty$ and has range $(-\infty, 0]$.

The right branch decreases from $+\infty$ to 0 and has range $(0, \infty)$.

These ranges are disjoint, so f is one-to-one on its entire domain and its range is $\mathbb{R}$.

Inverse. Write $y = \sqrt{x+4}/(x-1)$. Squaring gives

$$y^2 x^2 - (2y^2 + 1)x + y^2 - 4 = 0.$$

For $y \neq 0$, the quadratic formula gives

$$x = \frac{2y^2 + 1 \pm \sqrt{1 + 20y^2}}{2y^2}.$$

The original, unsquared equation requires $x < 1$ when $y < 0$ and $x > 1$ when $y > 0$.

Also $y = 0$ gives $x = -4$. Thus

$$f^{-1}(y) = \begin{cases} \frac{2y^2 + 1 - \sqrt{1 + 20y^2}}{2y^2}, & y < 0, \\ -4, & y = 0, \\ \frac{2y^2 + 1 + \sqrt{1 + 20y^2}}{2y^2}, & y > 0. \end{cases}$$

The sign choice restores the original equation after squaring. The inverse has domain $\mathbb{R}$ and range D .