

Question 1

Let $f(x) = \sqrt{x+4}$ and $g(x) = \frac{1}{x-2}$. Define the composition $h(x) = f(g(x))$.

Find the domain of $h(x)$, and express your answer in interval notation.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}, \quad g(x) = \frac{1}{x-2}, \quad h(x) = f(g(x)) = \sqrt{\frac{1}{x-2} + 4}$$

To determine the domain of $h(x)$, we consider the restrictions from both the inner and outer functions.

Step 1: Domain restriction from $g(x)$

The function $g(x) = \frac{1}{x-2}$ is undefined when the denominator is zero. This occurs at $x = 2$, so:

$$x \neq 2$$

Step 2: Domain restriction from the square root

We need the expression inside the square root to be greater than or equal to zero:

$$\frac{1}{x-2} + 4 \geq 0 \Rightarrow \frac{4x-7}{x-2} \geq 0$$

Step 3: Critical points and sign analysis

$$4x - 7 = 0 \Rightarrow x = \frac{7}{4}, \quad x - 2 = 0 \Rightarrow x = 2$$

From the sign chart:

Positive on $(-\infty, \frac{7}{4}]$, Negative on $(\frac{7}{4}, 2)$, Positive again on $(2, \infty)$.

Final Domain:

$$\boxed{(-\infty, \frac{7}{4}] \cup (2, \infty)}$$

Question 2

A piecewise function is defined as follows:

$$f(x) = \begin{cases} x^2 - 4x + 3 & \text{if } x < 2 \\ \sqrt{x+2} & \text{if } x \geq 2 \end{cases}$$

- (a) Determine whether $f(x)$ is continuous at $x = 2$.
- (b) Determine whether $f(x)$ is differentiable at $x = 2$.

Question 2 - Solution

Part (a): Continuity at $x = 2$ We must check:

- $\lim_{x \rightarrow 2^-} f(x)$
- $\lim_{x \rightarrow 2^+} f(x)$
- $f(2)$

For $x < 2$, use $f(x) = x^2 - 4x + 3$:

$$\lim_{x \rightarrow 2^-} f(x) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

For $x \geq 2$, use $f(x) = \sqrt{x+2}$:

$$\lim_{x \rightarrow 2^+} f(x) = \sqrt{2+2} = \sqrt{4} = 2$$

Since:

$$\lim_{x \rightarrow 2^-} f(x) = -1, \quad \lim_{x \rightarrow 2^+} f(x) = 2$$

the limit from left and right are not equal.

Conclusion: $f(x)$ is **not continuous** at $x = 2$

Part (b): Differentiability at $x = 2$

A function must be continuous at a point to be differentiable there. Since $f(x)$ is not continuous at $x = 2$, it cannot be differentiable there.

Conclusion: $f(x)$ is **not differentiable** at $x = 2$

Question 3

Let the function f be defined by

$$f(x) = \cos(\sin x)$$

- (a) Determine the period of $f(x)$.
- (b) Prove that $f(x)$ is an even function.
- (c) Find all values of $x \in [0, 2\pi]$ such that $f(x) = 1$.

Question 3 - Solution

(a) Although $\sin x$ has period 2π , the function

$$f(x) = \cos(\sin x)$$

has a smaller period because cosine is even.

Using

$$\sin(x + \pi) = -\sin x$$

and

$$\cos(-u) = \cos(u),$$

we get

$$f(x + \pi) = \cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore, $f(x)$ repeats after π . The reason the period changes from 2π to π is that shifting by π changes $\sin x$ to $-\sin x$, but cosine gives the same value for opposite inputs.

Thus, the period is

$$\boxed{\pi}.$$

(b) To prove that $f(x)$ is even, compute $f(-x)$:

$$f(-x) = \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore,

$$\boxed{f(x) \text{ is even.}}$$

(c) Solve

$$\cos(\sin x) = 1.$$

Cosine equals 1 when its input is $2k\pi$. Thus,

$$\sin x = 2k\pi.$$

But since $-1 \leq \sin x \leq 1$, the only possible value is

$$\sin x = 0.$$

On $[0, 2\pi]$, this occurs at

$$x = 0, \pi, 2\pi.$$

Therefore,

$$\boxed{x = 0, \pi, 2\pi}.$$

Question 4

$$\text{Let } f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ c, & x = 3 \end{cases}$$

- (a) Simplify $f(x)$ for $x \neq 3$.
- (b) Find the value of c that makes $f(x)$ continuous at $x = 3$.
- (c) Is $f(x)$ differentiable at $x = 3$? Justify your answer.

Question 4 - Solution

(a) Simplifying for $x \neq 3$:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3, \quad x \neq 3$$

So the simplified function is:

$$f(x) = x + 3 \quad \text{for } x \neq 3$$

(b) **Continuity at $x = 3$:**

To ensure continuity at $x = 3$, we must have:

$$\lim_{x \rightarrow 3} f(x) = f(3) = c$$

From (a), since $f(x) = x + 3$ for $x \neq 3$, then:

$$\lim_{x \rightarrow 3} f(x) = 3 + 3 = 6 \Rightarrow c = \boxed{6}$$

(c) **Differentiability at $x = 3$:**

We now check if $f(x)$ is differentiable at $x = 3$. The simplified form $f(x) = x + 3$ is linear and differentiable for all $x \neq 3$.

If we define $f(3) = 6$, then the function becomes:

$$f(x) = \begin{cases} x + 3 & x \neq 3 \\ 6 & x = 3 \end{cases}$$

This function is: Continuous at $x = 3$, Equal to $x + 3$ everywhere except possibly at 3, $f(x)$ approaches $x + 3$ smoothly at 3

So,

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3 + h + 3) - 6}{h} = \lim_{h \rightarrow 0} \frac{6 + h - 6}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 3$, and $f'(3) = \boxed{1}$

Question 5

A function f is defined as:

$$f(x) = \begin{cases} \frac{\ln(1+ax)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

- (a) Find the value of a for which $f(x)$ is continuous at $x = 0$.
- (b) For that value of a , determine whether f is differentiable at $x = 0$.

Question 5 - Solution

(a) For f to be continuous at $x = 0$, we need

$$\lim_{x \rightarrow 0} f(x) = f(0) = 1.$$

For $x \neq 0$,

$$f(x) = \frac{\ln(1 + ax)}{x}.$$

So compute

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x}.$$

As $x \rightarrow 0$, the numerator and denominator both approach 0, so we use L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x} = \lim_{x \rightarrow 0} \frac{\frac{a}{1+ax}}{1} = a.$$

For continuity, this limit must equal $f(0) = 1$. Therefore,

$$a = 1.$$

Thus,

$$\boxed{a = 1}.$$

(b) Now let $a = 1$. Then

$$f(x) = \frac{\ln(1 + x)}{x}, \quad x \neq 0,$$

and $f(0) = 1$.

To check differentiability at $x = 0$, use the definition:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}.$$

Substitute $f(h) = \frac{\ln(1 + h)}{h}$ and $f(0) = 1$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{\ln(1+h)}{h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\ln(1 + h) - h}{h^2}.$$

This is also a 0/0 form, so apply L'Hospital's Rule:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{2h} = \lim_{h \rightarrow 0} \frac{-h}{2h(1 + h)} = \lim_{h \rightarrow 0} \frac{-1}{2(1 + h)} = -\frac{1}{2}.$$

Since this limit exists, f is differentiable at $x = 0$, and

$$\boxed{f'(0) = -\frac{1}{2}}.$$

Question 6

Let $f(x) = |x - 2| + |x + 1|$.

- (a) Express $f(x)$ as a piecewise-defined function.
- (b) Sketch the graph of $f(x)$.
- (c) Determine all values of x where $f(x)$ is not differentiable. Justify your answer.

Question 6 - Solution

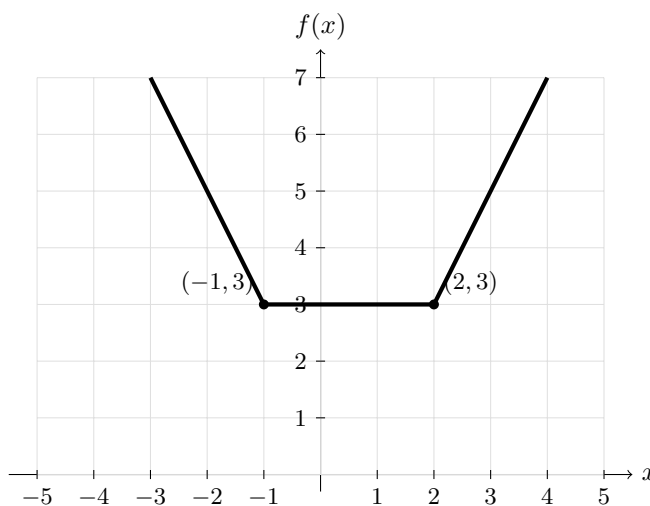
(a) The absolute values change sign at $x = -1$ and $x = 2$.

$$f(x) = \begin{cases} -(x-2) - (x+1), & x < -1, \\ -(x-2) + (x+1), & -1 \leq x < 2, \\ (x-2) + (x+1), & x \geq 2. \end{cases}$$

After simplifying,

$$f(x) = \begin{cases} -2x + 1, & x < -1, \\ 3, & -1 \leq x < 2, \\ 2x - 1, & x \geq 2. \end{cases}$$

(b) The graph decreases until $x = -1$, stays constant from $x = -1$ to $x = 2$, and then increases.



(c) The only possible non-differentiable points are where the absolute value expressions change sign:

$$x = -1 \quad \text{and} \quad x = 2.$$

From the piecewise function, the slopes are

$$\begin{cases} -2, & x < -1, \\ 0, & -1 < x < 2, \\ 2, & x > 2. \end{cases}$$

At $x = -1$, the slope changes from -2 to 0 , so f is not differentiable there.

At $x = 2$, the slope changes from 0 to 2 , so f is not differentiable there.

Therefore,

$$\boxed{x = -1, 2}$$

Question 7

Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

- (a) Determine all vertical and horizontal asymptotes of $f(x)$.
- (b) Identify any holes in the graph of $f(x)$, and specify their coordinates.
- (c) State the domain of $f(x)$.

Question 7 - Solution

We are given

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

Factor the numerator and denominator:

$$f(x) = \frac{(x+3)(x-1)}{(x-2)(x+2)}.$$

(a) Vertical asymptotes occur where the denominator is zero and no factor cancels. Since

$$x^2 - 4 = (x-2)(x+2),$$

the denominator is zero at

$$x = 2 \quad \text{and} \quad x = -2.$$

There are no common factors with the numerator, so both are vertical asymptotes:

$$x = -2, \quad x = 2.$$

Since the numerator and denominator have the same degree, the horizontal asymptote is the ratio of leading coefficients:

$$y = \frac{1}{1} = 1.$$

Thus,

$$y = 1.$$

(b) Holes occur when a factor cancels from the numerator and denominator. Since

$$\frac{(x+3)(x-1)}{(x-2)(x+2)}$$

has no common factors, there are

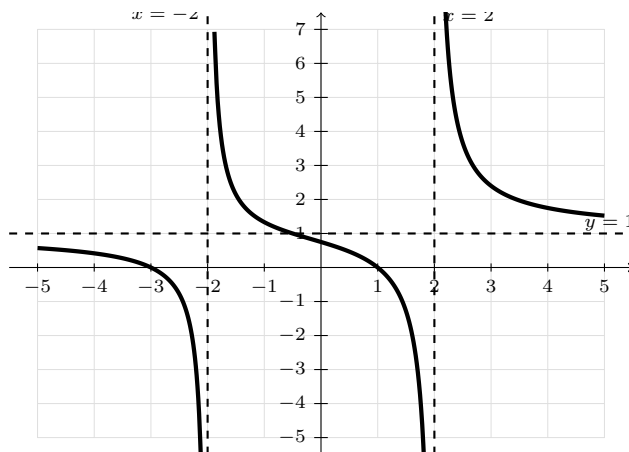
$$\text{no holes}.$$

(c) The domain excludes the values that make the denominator zero:

$$x \neq -2, \quad x \neq 2.$$

Therefore,

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$



Question 8

Let

$$f(x) = \frac{1}{x}, \quad g(x) = \sqrt{x-2}.$$

- (a) Find the composite function $(f \circ g)(x)$ and state its domain.
- (b) Determine whether $(f \circ g)(x)$ is defined at $x = 2$. Justify your answer.
- (c) Evaluate $(f \circ g)(6)$.

Question 8 - Solution**(a) Composite function and domain**

The composite function is

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-2}) = \frac{1}{\sqrt{x-2}}.$$

To determine the domain:

- The square root requires $x - 2 \geq 0 \Rightarrow x \geq 2$.
- The denominator cannot be zero, so $\sqrt{x-2} \neq 0 \Rightarrow x \neq 2$.

Thus, the domain is

$$\boxed{(2, \infty)}.$$

(b) Endpoint analysis

At $x = 2$,

$$(f \circ g)(2) = \frac{1}{\sqrt{2-2}} = \frac{1}{0},$$

which is undefined. Therefore, $x = 2$ is excluded from the domain.

(c) Evaluation

Evaluate the composite at $x = 6$:

$$(f \circ g)(6) = \frac{1}{\sqrt{6-2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Answer:

$$\boxed{\frac{1}{2}}$$

Question 9

Let $f(x) = \begin{cases} 2x + 5 & \text{if } x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$

- (a) Is $f(x)$ continuous at $x = 1$? Justify analytically.
- (b) Is $f(x)$ differentiable at $x = 1$? If not, explain why.

Question 9 - Solution**(a) Continuity at $x = 1$:**

Check:

$$\lim_{x \rightarrow 1^-} f(x) = 2(1) + 5 = 7$$

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 + 2 = 3$$

Since:

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x),$$

The two-sided limit does not exist, so:

$f(x)$ is not continuous at $x = 1$

(b) Differentiability at $x = 1$:

A function cannot be differentiable at a point where it is not continuous.

So:

$f(x)$ is not differentiable at $x = 1$

Explanation: The left and right-hand limits do not match at $x = 1$, so there is a jump discontinuity.

Question 10

Let the function f be defined by:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ c, & x = 2 \end{cases}$$

- (a) Find the limit $\lim_{x \rightarrow 2} f(x)$.
- (b) Determine the value of c that makes $f(x)$ continuous at $x = 2$.
- (c) Is $f(x)$ differentiable at $x = 2$ for the value of c you found? Justify your answer.

Question 10 - Solution

(a) Compute $\lim_{x \rightarrow 2} f(x)$:

Factor the numerator:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}, \quad x \neq 2$$

Cancel the common factor:

$$f(x) = x + 2 \quad \text{for } x \neq 2$$

So:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

Answer: $\boxed{4}$

(b) To make f continuous at $x = 2$, set:

$$f(2) = \lim_{x \rightarrow 2} f(x) = 4 \Rightarrow c = \boxed{4}$$

(c) **Differentiability at $x = 2$:**

With $c = 4$, we now have:

$$f(x) = \begin{cases} x + 2, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

The function is: Continuous at $x = 2$, Equal to a linear function near $x = 2$

Compute derivative using definition:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{(2 + h + 2) - 4}{h} = \frac{4 + h - 4}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 2$, with $f'(2) = \boxed{1}$