

Question 9

Suppose that f is continuous on an interval I and that

$$\int f(x) dx = F(x) + C.$$

Show that if two antiderivatives of f agree at a single point, then they are identical on the entire interval.

Question 9 - Solution

Let F and G be two antiderivatives of f on the interval I . Then

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Assume that there exists a point $x_0 \in I$ such that

$$F(x_0) = G(x_0).$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, H has zero derivative on I , so H is constant on I . That is,

$$H(x) = K \quad \text{for all } x \in I.$$

Evaluate H at x_0 :

$$H(x_0) = F(x_0) - G(x_0) = 0.$$

Hence, $K = 0$, and therefore

$$F(x) - G(x) = 0 \quad \text{for all } x \in I.$$

This implies

$$F(x) = G(x) \quad \text{for all } x \in I.$$

$$\boxed{F(x) \equiv G(x)}$$