

Question 4

Suppose that f is continuous on an interval I and that

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Show that every solution of the differential equation

$$y' = f(x)$$

has the form

$$y = F(x) + C$$

for some constant C .

Question 4 - Solution

Let y be any solution of the differential equation

$$y' = f(x)$$

on the interval I .

Since $F'(x) = f(x)$ and $y'(x) = f(x)$ for all $x \in I$, we have

$$y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = y(x) - F(x).$$

Differentiate $H(x)$:

$$H'(x) = y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Thus, $H'(x) = 0$ on I .

By the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back gives

$$y(x) - F(x) = C.$$

Rewriting,

$$y = F(x) + C.$$

$$\boxed{y = F(x) + C}$$