

Question 3

Let f be a continuous function on an interval I . Suppose that

$$\int f(x) \, dx = F(x) + C_1 \quad \text{and} \quad \int f(x) \, dx = G(x) + C_2$$

are two indefinite integrals of f . Prove that

$$F(x) - G(x) = \text{constant}.$$

Question 3 - Solution

Since both expressions represent antiderivatives of f , we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the difference of the two functions:

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I .

From the Mean Value Theorem, a function whose derivative is zero everywhere on an interval must be constant. Therefore, there exists a real number K such that

$$H(x) = K \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = K.$$

$F(x) - G(x) = \text{constant}$