

Question 10

Prove that for any positive integer n ,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

Question 10 - Solution

Write both sums explicitly:

$$\sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$
$$\sum_{k=1}^n \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}.$$

Subtract the second sum from the first:

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

All common terms cancel:

$$\frac{1}{2} - \frac{1}{2} = 0, \quad \frac{1}{3} - \frac{1}{3} = 0, \quad \dots$$

Only the first term of the first sum and the last term of the second sum remain:

$$1 - \frac{1}{n+1}.$$

Thus,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

$$\boxed{1 - \frac{1}{n+1}}$$