

Question 4

Prove that for any positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Question 4 - Solution

We prove the identity using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 (2k - 1) = 2(1) - 1 = 1,$$

and

$$1^2 = 1.$$

Thus, the statement holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^n (2k - 1) + (2(n + 1) - 1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} (2k - 1) = n^2 + (2n + 1).$$

Simplify:

$$n^2 + 2n + 1 = (n + 1)^2.$$

Thus,

$$\sum_{k=1}^{n+1} (2k - 1) = (n + 1)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the identity holds for all positive integers n .

$$\boxed{\sum_{k=1}^n (2k - 1) = n^2}$$