

Question 3

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Question 3 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^3 = 1^3 = 1, \quad \left(\frac{1(1+1)}{2} \right)^2 = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3.$$

Factor out $(n+1)^2$:

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1 \right).$$

Combine terms:

$$\frac{n^2 + 4n + 4}{4} = \frac{(n+2)^2}{4}.$$

Thus,

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \cdot \frac{(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2} \right)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2}$$