

Question 2

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Question 2 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^2 = 1^2 = 1,$$

and

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^n k^2 + (n+1)^2.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k^2 = (n+1) \left(\frac{n(2n+1)}{6} + (n+1) \right).$$

Combine terms inside the parentheses:

$$\frac{n(2n+1) + 6(n+1)}{6} = \frac{2n^2 + 7n + 6}{6} = \frac{(2n+3)(n+2)}{6}.$$

Thus,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}}$$