

Question 10

Prove that

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x = \frac{1}{e}.$$

Question 10 - Solution

Let

$$L = \lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x.$$

Take the natural logarithm:

$$\ln L = \lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+1} \right).$$

Rewrite the logarithm:

$$\ln \left(\frac{x}{x+1} \right) = \ln \left(\frac{1}{1 + \frac{1}{x}} \right) = -\ln \left(1 + \frac{1}{x} \right).$$

Thus,

$$\ln L = - \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Consider the limit

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Rewrite it as a quotient:

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}}.$$

This is an indeterminate form $\frac{0}{0}$, so apply L'Hôpital's Rule.

Differentiate numerator and denominator:

$$\frac{d}{dx} \ln \left(1 + \frac{1}{x} \right) = \frac{-1}{x(x+1)}, \quad \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{-1}{x(x+1)}}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1.$$

Therefore,

$$\ln L = -1.$$

Exponentiating both sides:

$$L = e^{-1} = \frac{1}{e}.$$

$$\boxed{\frac{1}{e}}$$