

Question 7

Prove that

$$\lim_{x \rightarrow \infty} x(\sqrt{x^2 + 1} - x) = \frac{1}{2}.$$

Question 7 - Solution

Begin with the expression

$$x(\sqrt{x^2 + 1} - x).$$

Factor the expression inside the parentheses using a conjugate:

$$\sqrt{x^2 + 1} - x = \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x} = \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \frac{1}{\sqrt{x^2 + 1} + x}.$$

Substitute this back:

$$x(\sqrt{x^2 + 1} - x) = \frac{x}{\sqrt{x^2 + 1} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + 1} = x\sqrt{1 + \frac{1}{x^2}}.$$

Thus,

$$\frac{x}{x\sqrt{1 + \frac{1}{x^2}} + x} = \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$