

Question 6

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 4x + 1} - x \right) = 2.$$

Question 6 - Solution

Start with the expression

$$\sqrt{x^2 + 4x + 1} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + 4x + 1} - x = \frac{(\sqrt{x^2 + 4x + 1} - x)(\sqrt{x^2 + 4x + 1} + x)}{\sqrt{x^2 + 4x + 1} + x}.$$

Simplify the numerator:

$$(x^2 + 4x + 1) - x^2 = 4x + 1.$$

Thus,

$$\sqrt{x^2 + 4x + 1} - x = \frac{4x + 1}{\sqrt{x^2 + 4x + 1} + x}.$$

Factor x from the square root in the denominator:

$$\sqrt{x^2 + 4x + 1} = x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}}.$$

Substitute:

$$\frac{4x + 1}{x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + x} = \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1}.$$

Take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1} = \frac{4}{1 + 1} = 2.$$

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