

Question 5

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x} - \sqrt{x^2 + x} \right) = 2.$$

Question 5 - Solution

We simplify the expression by factoring out x from each square root.

Consider

$$\sqrt{x^2 + 5x} - \sqrt{x^2 + x}.$$

Factor x^2 inside each square root:

$$\sqrt{x^2 \left(1 + \frac{5}{x}\right)} - \sqrt{x^2 \left(1 + \frac{1}{x}\right)}.$$

Since $x > 0$ for large x , this becomes

$$x \left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}} \right).$$

Multiply and divide by the conjugate:

$$x \cdot \frac{\left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}}\right) \left(\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}\right)}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Simplify the numerator:

$$\left(1 + \frac{5}{x}\right) - \left(1 + \frac{1}{x}\right) = \frac{4}{x}.$$

Thus the expression becomes

$$x \cdot \frac{\frac{4}{x}}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{1 + 1} = 2.$$

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