

Question 3

Prove that

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x = \frac{1}{2}.$$

Question 3 - Solution

We begin by algebraically simplifying the expression.

Consider

$$\sqrt{x^2 + x} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + x} - x = \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x}.$$

Simplify the numerator:

$$(x^2 + x) - x^2 = x.$$

Thus,

$$\sqrt{x^2 + x} - x = \frac{x}{\sqrt{x^2 + x} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + x} = x\sqrt{1 + \frac{1}{x}}.$$

Substitute:

$$\frac{x}{x\sqrt{1 + \frac{1}{x}} + x} = \frac{x}{x\left(\sqrt{1 + \frac{1}{x}} + 1\right)}.$$

Cancel x :

$$\frac{1}{\sqrt{1 + \frac{1}{x}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{\sqrt{1 + 0} + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$