

Question 2

Prove that

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0 \quad \text{for any positive integer } n.$$

Question 2 - Solution

We compare the growth rates of the polynomial x^n and the exponential function e^x .

Both x^n and e^x are differentiable for all real x . Apply L'Hôpital's Rule repeatedly to the limit

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x}.$$

Differentiate the numerator and denominator once:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(e^x) = e^x.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{nx^{n-1}}{e^x}.$$

Apply L'Hôpital's Rule again. After n applications, the numerator becomes a constant:

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x}.$$

Since e^x grows without bound as $x \rightarrow \infty$,

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

$$\boxed{0}$$