

Question 10

Prove that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3.$$

Question 10 - Solution

We derive the formula using the method of cross-sectional disks.

Place the sphere of radius R so that it is centered at the origin. Its equation is

$$x^2 + y^2 + z^2 = R^2.$$

Fix a value of x with $-R \leq x \leq R$. A cross-section perpendicular to the x -axis is a circle whose radius depends on x .

Solving the equation of the sphere for $y^2 + z^2$ gives

$$y^2 + z^2 = R^2 - x^2.$$

Thus, the radius of the cross-sectional disk at position x is

$$r(x) = \sqrt{R^2 - x^2}.$$

The area of this circular cross-section is

$$A(x) = \pi[r(x)]^2 = \pi(R^2 - x^2).$$

An infinitesimally thin slice of thickness dx has volume

$$dV = A(x) dx = \pi(R^2 - x^2) dx.$$

Integrate from $x = -R$ to $x = R$ to obtain the total volume:

$$V = \int_{-R}^R \pi(R^2 - x^2) dx.$$

Factor out π :

$$V = \pi \int_{-R}^R (R^2 - x^2) dx.$$

Evaluate the integral:

$$\int_{-R}^R R^2 dx = 2R^3, \quad \int_{-R}^R x^2 dx = \frac{2R^3}{3}.$$

Thus,

$$\int_{-R}^R (R^2 - x^2) dx = 2R^3 - \frac{2R^3}{3} = \frac{4R^3}{3}.$$

Substitute back:

$$V = \pi \cdot \frac{4R^3}{3} = \frac{4}{3}\pi R^3.$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$