

Question 9

Prove that the volume of a right circular cone with base radius R and height h is

$$V = \frac{1}{3}\pi R^2 h.$$

Question 9 - Solution

We derive the formula using cross-sectional disks and integration.

Place the cone so that its tip is at the origin and its base lies in the plane $x = h$. The axis of the cone lies along the x -axis.

At a distance x from the tip, where $0 \leq x \leq h$, the radius of the cone is determined by similar triangles.

Since the radius grows linearly from 0 to R over the height h , the radius at position x is

$$r(x) = \frac{R}{h}x.$$

A cross-section perpendicular to the x -axis at position x is a circle of radius $r(x)$. Its area is

$$A(x) = \pi[r(x)]^2 = \pi \left(\frac{R}{h}x \right)^2 = \pi \frac{R^2}{h^2}x^2.$$

An infinitesimal slice of thickness dx has volume

$$dV = A(x) dx = \pi \frac{R^2}{h^2}x^2 dx.$$

Integrate from $x = 0$ to $x = h$ to obtain the total volume:

$$V = \int_0^h \pi \frac{R^2}{h^2}x^2 dx.$$

Factor out the constants:

$$V = \pi \frac{R^2}{h^2} \int_0^h x^2 dx.$$

Evaluate the integral:

$$\int_0^h x^2 dx = \left[\frac{x^3}{3} \right]_0^h = \frac{h^3}{3}.$$

Substitute back:

$$V = \pi \frac{R^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi R^2 h.$$

$$\boxed{V = \frac{1}{3}\pi R^2 h}$$