

Question 7

Prove that the area of a circle of radius R is

$$A = \pi R^2.$$

Question 7 - Solution

We derive the area formula using integration.

Consider a circle of radius R centered at the origin. Its equation is

$$x^2 + y^2 = R^2.$$

Solving for y gives the upper semicircle:

$$y = \sqrt{R^2 - x^2}.$$

The area of the entire circle is twice the area of the upper semicircle:

$$A = 2 \int_{-R}^R \sqrt{R^2 - x^2} dx.$$

To evaluate the integral, use the substitution

$$x = R \sin \theta, \quad dx = R \cos \theta d\theta.$$

When $x = -R$, $\theta = -\frac{\pi}{2}$, and when $x = R$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} \sqrt{R^2 - R^2 \sin^2 \theta} (R \cos \theta) d\theta.$$

Simplify:

$$\sqrt{R^2(1 - \sin^2 \theta)} = R \cos \theta.$$

Thus,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta = \frac{R^2}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta)) d\theta.$$

Evaluate:

$$\int_{-\pi/2}^{\pi/2} 1 d\theta = \pi, \quad \int_{-\pi/2}^{\pi/2} \cos(2\theta) d\theta = 0.$$

So,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \frac{R^2}{2} \pi.$$

Finally, multiply by 2 to get the area of the full circle:

$$A = 2 \cdot \frac{R^2}{2} \pi = \pi R^2.$$

$$\boxed{A = \pi R^2}$$