

Question 6

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

Question 6 - Solution

We derive the formula using washers (disks with holes).

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval near x_i^* , the region between the curves has an outer radius

$$R_i = f(x_i^*)$$

and an inner radius

$$r_i = g(x_i^*).$$

When this vertical strip is revolved about the x -axis, it forms a thin washer.

The area of the outer disk is

$$\pi R_i^2 = \pi [f(x_i^*)]^2,$$

and the area of the inner disk (the hole) is

$$\pi r_i^2 = \pi [g(x_i^*)]^2.$$

Thus, the area of the washer is

$$\pi ([f(x_i^*)]^2 - [g(x_i^*)]^2).$$

Multiplying by the thickness Δx_i , the volume of the washer is

$$\Delta V_i = \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

Adding the volumes of all washers gives an approximation to the total volume:

$$\sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, the washers fill the solid more accurately. Since f and g are continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$