

Question 5

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 5 - Solution

We derive the formula using the idea of approximating the solid with thin disks.

Partition the interval $[a, b]$ into subintervals

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ is approximately constant at height $f(x_i^*)$. When this vertical strip is revolved about the x -axis, it forms a thin circular disk.

The radius of the disk is

$$r_i = f(x_i^*).$$

The area of the circular face of the disk is

$$A_i = \pi r_i^2 = \pi [f(x_i^*)]^2.$$

Since the thickness of the disk is Δx_i , its volume is approximately

$$\Delta V_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

Adding the volumes of all disks gives an approximation to the total volume:

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and the maximum subinterval length approaches zero, these disks more accurately fill the solid. Because f is continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

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