

Question 4

Assume $0 \leq a < b$ and that f is continuous and nonnegative on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the y -axis is

$$V = 2\pi \int_a^b x f(x) dx.$$

Question 4 - Solution

Take a partition $a = x_0 < \cdots < x_n = b$ and let m_i, M_i be the minimum and maximum of f on $[x_{i-1}, x_i]$.

Since $a \geq 0$, the annular cylinders occupy disjoint radial intervals. Their volumes bound the desired volume:

$$\sum_i \pi(x_i^2 - x_{i-1}^2)m_i \leq V \leq \sum_i \pi(x_i^2 - x_{i-1}^2)M_i.$$

Let $c_i = (x_i + x_{i-1})/2$. Then $\pi(x_i^2 - x_{i-1}^2) = 2\pi c_i \Delta x_i$.

Uniform continuity of f makes $\max_i (M_i - m_i) \rightarrow 0$ as the mesh tends to zero, so the difference between the upper and lower bounds tends to zero.

Both bounds approach the Riemann integral of $2\pi x f(x)$. Hence

$$V = 2\pi \int_a^b x f(x) dx.$$