

Question 3

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b].$$

Prove that the area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b (f(x) - g(x)) \, dx.$$

Question 3 - Solution

We approximate the region using vertical rectangles.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

At the point x_i^* , the height of the region between the curves is

$$f(x_i^*) - g(x_i^*).$$

Thus, the area of the corresponding rectangle is

$$\Delta A_i = (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

The total area of all rectangles is approximated by the sum

$$\sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, these rectangles more accurately approximate the region between the curves. Since f and g are continuous, the limit of the sums exists.

Taking the limit gives

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i = \int_a^b (f(x) - g(x)) \, dx.$$

$$A = \int_a^b (f(x) - g(x)) \, dx$$