

Question 2

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 2 - Solution

We approximate the solid of revolution using thin disks.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval.

Over the interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ generates a disk when revolved about the x -axis.

The radius of this disk is

$$r_i = f(x_i^*).$$

Thus, the volume of the disk is

$$\Delta V_i = \pi r_i^2 \Delta x_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

The total volume of all disks is approximated by the sum

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, this sum approaches the exact volume of the solid. Since f is continuous, the limit exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

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