

Question 1

Assume that f is a continuous, nonnegative function on the interval $[a, b]$. Prove that the area of the region bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) \, dx.$$

Question 1 - Solution

We interpret area using approximating rectangles.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Since $f(x) \geq 0$, the area of the rectangle over $[x_{i-1}, x_i]$ with height $f(x_i^*)$ is

$$\text{Area}_i = f(x_i^*) \Delta x_i.$$

The total area of all rectangles is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

As the partition is refined (that is, as $\|P\| \rightarrow 0$), the rectangles better approximate the region under the curve. Because f is continuous on $[a, b]$, it is integrable, and the limit of these sums exists.

Taking the limit gives

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

By definition, this limit equals the exact area of the region under the curve.

Therefore, the area of the region bounded by $y = f(x)$, the x -axis, and $x = a$, $x = b$ is

$$A = \int_a^b f(x) dx.$$

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