

Question 10

Assume that f is integrable on $[a, b]$ and that there exists a real number M such that

$$f(x) \leq M \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq M(b - a).$$

Question 10 - Solution

Define the function

$$h(x) = M - f(x).$$

Since f is integrable on $[a, b]$ and M is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq M$ for all $x \in [a, b]$, we have

$$h(x) = M - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (M - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b M \, dx - \int_a^b f(x) \, dx \geq 0.$$

Since

$$\int_a^b M \, dx = M(b - a),$$

we obtain

$$M(b - a) - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq M(b - a).$$

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