

Question 9

Assume that f is integrable on $[a, b]$ and that there exists a real number m such that

$$f(x) \geq m \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq m(b - a).$$

Question 9 - Solution

Define a new function

$$h(x) = f(x) - m.$$

Since f is integrable on $[a, b]$ and m is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \geq m$ for all $x \in [a, b]$, we have

$$h(x) = f(x) - m \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (f(x) - m) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b f(x) \, dx - \int_a^b m \, dx \geq 0.$$

Since $\int_a^b m \, dx = m(b - a)$, we obtain

$$\int_a^b f(x) \, dx - m(b - a) \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \geq m(b - a).$$

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