

Question 8

Assume that f is integrable on $[a, b]$ and that c satisfies

$$a < c < b.$$

Prove that

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

Question 8 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_k = c < \cdots < x_n = b$$

be a partition of $[a, b]$ that includes the point c .

Let

$$\Delta x_i = x_i - x_{i-1}$$

and choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Split this sum at c :

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = \sum_{i=1}^k f(x_i^*) \Delta x_i + \sum_{i=k+1}^n f(x_i^*) \Delta x_i.$$

The first sum is a Riemann sum for f over $[a, c]$, and the second sum is a Riemann sum for f over $[c, b]$.

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f is integrable on $[a, b]$, it is integrable on both subintervals $[a, c]$ and $[c, b]$.

Thus,

$$\begin{aligned} \lim_{\|P\| \rightarrow 0} \sum_{i=1}^k f(x_i^*) \Delta x_i &= \int_a^c f(x) dx, \\ \lim_{\|P\| \rightarrow 0} \sum_{i=k+1}^n f(x_i^*) \Delta x_i &= \int_c^b f(x) dx. \end{aligned}$$

Therefore,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

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