

Question 7

Assume that f is integrable on $[a, b]$. Prove that

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

Question 7 - Solution

We begin with a basic inequality involving absolute values.

For all real numbers u , we have

$$-|u| \leq u \leq |u|.$$

Apply this inequality to $f(x)$:

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \text{for all } x \in [a, b].$$

Integrate each part of the inequality over $[a, b]$:

$$\int_a^b -|f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

Simplify the left-hand integral:

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

This shows that the value of $\int_a^b f(x) \, dx$ lies between $-\int_a^b |f(x)| \, dx$ and $\int_a^b |f(x)| \, dx$.

By definition of absolute value, this implies

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

$$\boxed{\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx}$$