

Question 6

Assume that f and g are integrable on $[a, b]$ and that

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

Question 6 - Solution

Define a new function

$$h(x) = g(x) - f(x).$$

Since f and g are integrable on $[a, b]$, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq g(x)$ for all $x \in [a, b]$, we have

$$h(x) = g(x) - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (g(x) - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b g(x) \, dx - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

$$\boxed{\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx}$$