

**Question 5**

Assume that  $f$  is integrable on  $[a, b]$  and that

$$f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq 0.$$

**Question 5 - Solution**

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval  $[a, b]$ , and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point  $x_i^*$  in each subinterval  $[x_{i-1}, x_i]$ .

A Riemann sum for  $f$  on  $[a, b]$  is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Since  $f(x) \geq 0$  for all  $x \in [a, b]$  and  $\Delta x_i > 0$ , each term in the sum satisfies

$$f(x_i^*) \Delta x_i \geq 0.$$

Thus, every Riemann sum for  $f$  over  $[a, b]$  is nonnegative:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \geq 0.$$

Now take the limit as the norm of the partition  $\|P\| \rightarrow 0$ . Since  $f$  is integrable, the limit of the Riemann sums exists and equals the definite integral:

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Because every sum in the limit is nonnegative, the limit itself must be nonnegative.

Therefore,

$$\int_a^b f(x) dx \geq 0.$$

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