

Question 4

Assume that f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$$

Question 4 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, with subinterval widths

$$\Delta x_i = x_i - x_{i-1}.$$

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now consider the interval $[b, a]$. Reverse the order of the partition:

$$b = x_n > x_{n-1} > \cdots > x_0 = a.$$

The corresponding subinterval widths are

$$\Delta x'_i = x_{i-1} - x_i = -\Delta x_i.$$

A Riemann sum for f over $[b, a]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x'_i = \sum_{i=1}^n f(x_i^*) (-\Delta x_i) = - \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Take the limit as the norm of the partition goes to zero. Since f is integrable on $[a, b]$, the limit exists:

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

Rewriting,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$\boxed{\int_a^b f(x) dx = - \int_b^a f(x) dx}$$