

Question 2

Assume that f and g are integrable on the interval $[a, b]$. Prove that

$$\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.$$

Question 2 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for $f + g$ is

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i.$$

Using algebra, split the sum:

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i = \sum_{i=1}^n f(x_i^*) \Delta x_i + \sum_{i=1}^n g(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f and g are integrable on $[a, b]$, the limits of both sums exist:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx,$$

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n g(x_i^*) \Delta x_i = \int_a^b g(x) dx.$$

Therefore,

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$$\boxed{\int_a^b (f + g) = \int_a^b f + \int_a^b g}$$