

Question 1

Assume that f is integrable on the interval $[a, b]$ and that c is a constant. Prove that

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx.$$

Question 1 - Solution

We prove this property using the definition of the definite integral in terms of Riemann sums.

Let P be a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

A Riemann sum for cf on $[a, b]$ is

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i.$$

Factor out the constant c :

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i = c \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$. Since f is integrable on $[a, b]$, the limit of its Riemann sums exists:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

Therefore,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n cf(x_i^*) \Delta x_i = c \int_a^b f(x) dx.$$

By definition of the definite integral, this limit equals

$$\int_a^b cf(x) dx.$$

Hence,

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx$$