

Question 9

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and that

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

Prove that f is constant on $[a, b]$.

Question 9 - Solution

Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We will show that

$$f(x_1) = f(x_2).$$

Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , the Mean Value Theorem applies. Thus, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) = 0$, so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$f(x_2) - f(x_1) = 0.$$

Therefore,

$$f(x_2) = f(x_1).$$

Because x_1 and x_2 were arbitrary points in $[a, b]$, we conclude that $f(x)$ has the same value at every point in the interval.

$$\boxed{f \text{ is constant on } [a, b]}$$