

Question 7

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave down on (a, b) .

Question 7 - Solution

To show that f is concave down on (a, b) , we must show that the derivative $f'(x)$ is decreasing on (a, b) .

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b) , the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) < 0.$$

Thus,

$$f'(x_2) < f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the derivative $f'(x)$ is decreasing on (a, b) .

Therefore, the function f is concave down on (a, b) .

$f \text{ is concave down on } (a, b)$