

Question 3

Assume that f is continuous on $[a, b]$ and differentiable on (a, b) . Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Question 3 - Solution

Define a new function

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) (x - a).$$

The function g is continuous on $[a, b]$ and differentiable on (a, b) because it is formed from functions with these properties.

Evaluate g at the endpoints:

$$g(a) = f(a) - \left(\frac{f(b) - f(a)}{b - a} \right) (a - a) = f(a),$$

$$g(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a} \right) (b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Thus,

$$g(a) = g(b).$$

By Rolle's Theorem, there exists a number $c \in (a, b)$ such that

$$g'(c) = 0.$$

Differentiate $g(x)$:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Substitute $x = c$:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Solve for $f'(c)$:

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