

Question 2

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and

$$f(a) = f(b).$$

Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = 0.$$

Question 2 - Solution

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem guarantees that f attains both a maximum and a minimum on $[a, b]$.

Let

$$M = \max_{x \in [a, b]} f(x), \quad m = \min_{x \in [a, b]} f(x).$$

If $M = m$, then f is constant on $[a, b]$, and hence

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

In particular, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Now assume that $M \neq m$. Since $f(a) = f(b)$, the maximum or minimum cannot occur at both endpoints.

Thus, at least one of the extreme values M or m must occur at some point $c \in (a, b)$.

At such a point c , the function f has either a local maximum or a local minimum. Because f is differentiable at c , the derivative must vanish there.

Therefore,

$$f'(c) = 0.$$

There exists $c \in (a, b)$ such that $f'(c) = 0$