

Question 1

Assume that f has a local maximum at $x = a$ and that f is differentiable at $x = a$. Prove that

$$f'(a) = 0.$$

Question 1 - Solution

Since f has a local maximum at $x = a$, there exists $\delta > 0$ such that

$$f(x) \leq f(a) \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Consider the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

for small values of h .

Case 1: $h > 0$

If $0 < h < \delta$, then $a + h$ lies within the interval where $f(a)$ is a local maximum. Hence,

$$f(a+h) - f(a) \leq 0.$$

Since $h > 0$, dividing by h preserves the inequality:

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Case 2: $h < 0$

If $-\delta < h < 0$, then again $a + h$ lies within the interval of the local maximum, so

$$f(a+h) - f(a) \leq 0.$$

Since $h < 0$, dividing by h reverses the inequality:

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Step 3: Take the limit

For $h > 0$, the difference quotient is less than or equal to 0. For $h < 0$, the difference quotient is greater than or equal to 0.

Since f is differentiable at a , the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists and must be the same from both sides.

The only number that is both ≤ 0 and ≥ 0 is 0.

Therefore,

$$f'(a) = 0.$$

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