

Question 10

Prove that

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x} = 0.$$

Question 10 - Solution

We begin by rewriting the expression to isolate a known limit.

Factor out x from the numerator:

$$\frac{\tan x - x}{x} = \frac{\tan x}{x} - 1.$$

Now rewrite $\tan x$ in terms of sine and cosine:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right).$$

Take limits of each factor as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1, \quad \text{so} \quad \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$$

Thus,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Finally,

$$\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} - 1 \right) = 1 - 1 = 0.$$

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