

Question 2

Prove that

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Question 2 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall the identity

$$1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right).$$

Substitute this into the limit:

$$\frac{1 - \cos x}{x} = \frac{2 \sin^2\left(\frac{x}{2}\right)}{x}.$$

Rewrite the fraction:

$$\frac{2 \sin^2\left(\frac{x}{2}\right)}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{2 \sin\left(\frac{x}{2}\right)}{x}.$$

Now rewrite the second factor:

$$\frac{2 \sin\left(\frac{x}{2}\right)}{x} = \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Thus,

$$\frac{1 - \cos x}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Now take limits of each factor as $x \rightarrow 0$.

Since $\lim_{u \rightarrow 0} \sin u = 0$, we have

$$\lim_{x \rightarrow 0} \sin\left(\frac{x}{2}\right) = 0.$$

Also, using the fundamental trig limit

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1,$$

we obtain

$$\lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1.$$

Therefore, by the product law for limits,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \cdot 1 = 0.$$

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