

Question 1

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Question 1 - Solution

We prove this limit using geometric inequalities.

Consider a unit circle centered at the origin.

Let $0 < x < \pi/2$ be measured in radians, and consider the angle x .

The area of the sector with angle x is

$$\text{Area of sector} = \frac{1}{2}x.$$

The area of the triangle formed by the radius and the angle x is

$$\text{Area of triangle} = \frac{1}{2} \sin x.$$

The area of the triangle formed by the tangent line is

$$\text{Area of outer triangle} = \frac{1}{2} \tan x.$$

Thus, the areas satisfy

$$\frac{1}{2} \sin x < \frac{1}{2}x < \frac{1}{2} \tan x.$$

Multiply through by 2:

$$\sin x < x < \tan x.$$

Divide all parts by $\sin x$ (which is positive for $0 < x < \pi/2$):

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals reverses the inequalities:

$$\cos x < \frac{\sin x}{x} < 1.$$

Now take the limit as $x \rightarrow 0^+$.

Since

$$\lim_{x \rightarrow 0} \cos x = 1,$$

the Squeeze Theorem gives

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1.$$

A similar argument applies for $x \rightarrow 0^-$, so the two-sided limit exists.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$