

Question 10

Assume that f is differentiable at $x = a$ and that g is differentiable at $x = f(a)$. Prove the Chain Rule:

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

Question 10 - Solution

Put $b = f(a)$. Differentiability of g at b means

$$g(b + u) - g(b) = g'(b)u + u\eta(u),$$

where $\eta(u) \rightarrow 0$ as $u \rightarrow 0$; define $\eta(0) = 0$ so this identity also holds at $u = 0$.

For $h \neq 0$, set $u = f(a + h) - f(a)$. Then

$$\frac{g(f(a + h)) - g(f(a))}{h} = (g'(b) + \eta(f(a + h) - f(a))) \frac{f(a + h) - f(a)}{h}.$$

Differentiability implies continuity of f at a , so the first factor tends to $g'(b)$ and the second to $f'(a)$.

$$\boxed{(g \circ f)'(a) = g'(f(a))f'(a).}$$

This argument remains valid when $f(a + h) = f(a)$ for some or all nearby h .