

Question 9

Prove from first principles that if

$$f(x) = x^3,$$

then

$$f'(a) = 3a^2.$$

Question 9 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^3$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}.$$

Expand the cube:

$$(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h}.$$

Factor out h :

$$f'(a) = \lim_{h \rightarrow 0} \frac{h(3a^2 + 3ah + h^2)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \rightarrow 0} (3a^2 + 3ah + h^2).$$

Now take the limit:

$$f'(a) = 3a^2.$$

Thus, using first principles,

$$\boxed{f'(a) = 3a^2}.$$