

Question 4

Assume that f is differentiable at $x = a$. Prove that f is continuous at $x = a$.

Question 4 - Solution

To show that f is continuous at $x = a$, we must prove that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Since f is differentiable at a , by definition,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Because this limit exists, the expression

$$\frac{f(a+h) - f(a)}{h}$$

remains bounded and approaches a finite value as $h \rightarrow 0$.

Rewrite $f(a+h) - f(a)$ as

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) h.$$

Now take limits as $h \rightarrow 0$:

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = \left(\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \rightarrow 0} h \right).$$

Since

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{and} \quad \lim_{h \rightarrow 0} h = 0,$$

we obtain

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = 0.$$

Therefore,

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

This shows that

$$\lim_{x \rightarrow a} f(x) = f(a),$$

so f is continuous at $x = a$.