

Question 4

Assume that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Prove the quotient rule:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Question 4 - Solution

By definition of the derivative,

$$\left(\frac{f}{g}\right)'(a) = \lim_{h \rightarrow 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h}.$$

Combine the fractions in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a) - f(a)g(a+h)}{h g(a+h)g(a)}.$$

Rewrite the numerator by adding and subtracting $f(a)g(a)$:

$$f(a+h)g(a) - f(a)g(a+h) = g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a)).$$

Substitute this expression:

$$= \lim_{h \rightarrow 0} \frac{g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a))}{h g(a+h)g(a)}.$$

Separate the fraction:

$$= \lim_{h \rightarrow 0} \left[\frac{g(a)}{g(a+h)g(a)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a+h)g(a)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Simplify coefficients:

$$= \lim_{h \rightarrow 0} \left[\frac{1}{g(a+h)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a)g(a+h)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Since g is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} g(a+h) = g(a).$$

Taking limits term by term,

$$\left(\frac{f}{g}\right)'(a) = \frac{1}{g(a)}f'(a) - \frac{f(a)}{[g(a)]^2}g'(a).$$

Combine the terms:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$