

Question 3

Assume that f and g are differentiable at $x = a$. Prove the product rule:

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Question 3 - Solution

By definition of the derivative,

$$(fg)'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Add and subtract $f(a+h)g(a)$ in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}.$$

Group the terms:

$$= \lim_{h \rightarrow 0} \left[\frac{f(a+h)(g(a+h) - g(a))}{h} + \frac{g(a)(f(a+h) - f(a))}{h} \right].$$

Separate the limit:

$$= \lim_{h \rightarrow 0} f(a+h) \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} + g(a) \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

Also, by differentiability,

$$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} = g'(a), \quad \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Substitute these limits:

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Thus,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$