

Question 10

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} f(x) = M.$$

Prove that

$$L = M.$$

Question 10 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L \neq M.$$

Then the positive number

$$\varepsilon = \frac{|L - M|}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

Similarly, since $\lim_{x \rightarrow a} f(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|f(x) - M| < \varepsilon.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x such that $0 < |x - a| < \delta$, both inequalities hold.

Using the triangle inequality,

$$|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M|.$$

Substitute the bounds:

$$|L - M| < \varepsilon + \varepsilon = 2\varepsilon = |L - M|.$$

This is a contradiction.

Therefore, our assumption that $L \neq M$ is false. We conclude that

$$L = M.$$