

Question 9

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = L.$$

Prove that

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$

Question 9 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - g(x)| < \varepsilon.$$

Begin by rewriting the difference:

$$f(x) - g(x) = (f(x) - L) - (g(x) - L).$$

Take absolute values and apply the triangle inequality:

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_1.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = L$, there exists $\delta_2 > 0$ such that

$$|g(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$,

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$